

# Depletion of energy from Naked Singular regions during gravitational collapse

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A distinguishable and observable physical property of Naked Singular Regions of the spacetime formed during a gravitational collapse has important implications for both experimental and theoretical relativity. We examine here whether energy can escape physically from naked singular regions to reach either a local or a distant observer within the framework of general relativity. We find that in case of imploding null dust collapse scenarios field outgoing singular null geodesics including the cauchy horizon can be immersed between two Vaidya spacetimes as null boundary layers with non vanishing positive energy density. Thus energy can be transported from the naked singularity to either a local or a distant observer. An example illustrating that similar considerations can be applied to dust models is given.

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## I. INTRODUCTION

A star with sufficient remnant mass ( $\gtrsim 3M_\odot$ ), on completion of its nuclear fuel cycle, must enter the phase of a continuous gravitational collapse. Once the nuclear fuel is exhausted gravitational forces become all powerful and hence star's internal pressure can not sustain the equilibrium resulting in a continued collapse [1, 2]. In the late stages of collapse the gravitational forces become dominant and the physics of collapse is determined mainly by the theory of general relativity. Under quite general and physical situations general relativity predicts that such a collapse must end in a singularity, i.e., a region of spacetime with extreme curvatures [3–5]. Physically one could describe singularity as a region of space with vanishing volume and unbounded gravitational forces. General relativity, however, does not say anything about the nature or physical properties of such a singularity. This is partially due to the fact that mathematical structure breaks down preventing analysis at and beyond the singularity. One could perhaps argue that as collapse progresses and matter is condensed in a region comparable to Planck length the quantum physical properties of spacetime would become dominant, thus preventing the formation of singularity. But this picture may not hold also since gravity as a force is very different in its nature in comparison to other forces and has a geometrical interpretation as curvature of spacetime. Moreover, despite numerous efforts, a viable quantum theory of gravity is not in sight. Hence for such regions of spacetime, whether relativity theory or quantum physics would determine the physics is still an open question.

To fill in the gap in our understanding of spacetime singularities in a mathematical consistent manner, a cos-

mic censorship conjecture, that all gravitational collapse must end in a black hole was proposed [3, 6]. The physical consequence of such a hypothesis is that even before the formation of a singularity a trapped surface develops covering the singularity from the outside world. Hence from a physical point of view singularity is hidden from the outside world. Initial studies in censorship were directed towards formulating the conjecture in a mathematically precise manner which could then possibly be proven [7]. This also led to formulation of other conjectures like, hoop conjecture by Kip Thorne and Siefert's conjecture [8, 9]. However, extensive studies in collapse with various forms of matter fields have shown that under fairly generic reasonable physical conditions both naked singularity and black holes would form as an end state of collapse, depending on various initial and boundary conditions [10]. It is still not very clear how to classify either matter or the initial and boundary conditions in a satisfactory way which would end in either state of singularity (naked or covered). Thus from the studies this far almost all physically reasonable matter fields lead to both naked and covered singularities during collapse (see, [11], and references therein).

Considerable work has since been done on naked singularities from the point of view of giving counterexamples to cosmic censorship but also on the study of their nature and structure. Having established their existence it is important to study the phenomena of formation of naked singularity from a more astrophysical perspective. One could look for a possible observable signature of naked singularity distinguishing them from other compact strong gravity objects, like black holes. In the studies carried out this far the stress has been towards showing that for a naked singularity to be “observable” a family of lightlike geodesics must terminate at the singularity [12, 13]. Optical appearance and redshift for such possible radiation has also been studied [14]. However, from the point of view of general relativity the first null ray coming out of singularity forms a Cauchy horizon (CH), and the spacetime model cannot remain valid after its

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formation. Therefore, without any consistent extension of spacetime beyond CH the validity and usefulness of all such geodesic analysis becomes doubtful. The basic question of the existence of the spacetime structure after the CH is unaddressed (it is difficult to provide extensions of spacetimes, for example, even for shell-crossing singularities which are gravitationally weak [15]) which is of utmost importance if we want to talk about families of geodesics ending at singularity in the past, making it a possible astrophysical source.

In this paper we wish to study the structure of the spacetime from this perspective. Is it possible to connect the two spacetimes before and after with Cauchy horizon as the boundary? Whether the resulting spacetime after the CH has formed can still have the same symmetry? Does relativity theory allows such continuation of spacetime through CH and whether boundary conditions pose any restrictions? Furthermore, can these boundary layers carry energy from naked singularity to a distant observer? Earlier Hiscock et al. has considered a model spacetime in which cauchy horizon ultimately becomes the event horizon of the schwarzschild black hole with non vanishing surface energy density and where it could be visible to observers falling into the blackhole [18].

If indeed the formation of a naked singularity is a physical phenomenon then the CH would represent a null surface layer emanating from the naked singularity, and reaching the distant observer separating the two spacetimes. It has been suggested in various studies that naked singularities may be responsible for various high energy phenomena in our universe (for example gamma ray bursts etc. [16]). It has also been suggested that in the late stages of collapse, when spacetime shrinks to size of the order of planck length quantum effects would play a dominant role resulting in either a burst of particle creation or preventing the formation of singularity all together [17]. Our aim in this paper is to examine two examples of naked singularities within the frame work of general relativity and whether this allows such a scenario as emission of a impulsive null wave carrying energy from the naked singularity. The result of such a study would have manifold implications. First does there exist a spacetime after the formation of a naked singularity which can be joined satisfactorily together with the original model separated by the null shockwave (CH)? If such a spacetime exists then whether it allows the existence of outgoing families of geodesics terminating at the singularity in the past. Second, and equally important, question is the structure of the CH itself. Whether this null surface ‘boundary layer’ is allowed to carry huge amounts of energy along the null ray to distant observer? And, if the answer is in affirmative, what is its structure and whether this scenario can be called a valid solution to the Einstein equations?

## II. A COLLAPSING STAR

Despite numerous exact solutions of the field equations, very few exact solutions of the field equations exist which can describe a physically reasonable collapsing matter cloud. In fact, in nearly all the studies of spherically symmetric collapse, the key models are either Lemaitre -Tolman-Bondi metric (LTB) [19] or the Vaidya spacetime [20]. Both these spacetimes have been well studied, and very well may be the only physically reasonable exact solution available. In all such studies it has been shown that there are out going null and time like geodesics which terminate at the naked singularity in the past. The visibility of the singularity in terms of a *roots equation* whose roots are tangents to the outgoing radial null geodesics with past end points at the singularity. Therefore, problem of relating initial data with end state of collapse is reduced to finding roots of a polynomial equation [13, 21].

We would first take up the Vaidya spacetime. Existence of naked singularity in this model is well established [22]. In particular for the case of a imploding shell with a linear mass function  $M(v) = \lambda v$ , for  $\lambda \leq 1/8$  singularity is known to be a naked singularity.

The metric describing a spherically symmetric Vaidya space-time is given by

$$ds^2 = - \left( 1 - \frac{2M(v)}{r} \right) dv^2 + 2dvdr + r^2 d\Omega^2, \quad (1)$$

where  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ . For linear mass function case  $2M(v) = \lambda v$ , and the singularity formed at  $v = 0$ ,  $r = 0$ , is naked iff:

$$x^2 - \frac{x}{\lambda} + \frac{2}{\lambda} = 0 \quad (2)$$

has real and positive root where  $x = \frac{v}{r}$ . It follows that for  $\lambda \leq 1/8$ , the above has two real and positive roots namely  $(\gamma, \beta)$ ,  $\beta > \gamma$  given by

$$\gamma = \frac{1 - \sqrt{1 - 8\lambda}}{2\lambda}, \quad \beta = \frac{1 + \sqrt{1 - 8\lambda}}{2\lambda}. \quad (3)$$

CH is the first null geodesic given by

$$v = \gamma r. \quad (4)$$

while a family of geodesics which terminate at the singularity in the past with the tangent  $x = \beta$  are given by

$$r = V \frac{(\beta - x)^{\frac{\gamma}{\beta - \gamma}}}{(x - \gamma)^{\frac{\beta}{\beta - \gamma}}}. \quad (5)$$

Where V is a parameter (constant along out going null geodesics) labeling different geodesics of the family.

Once the singularity forms the spacetime below the CH is described by the metric above. However, if the further

analysis of family of geodesics is to be valid than the spacetime beyond CH must also be described by similar metric with CH as the boundary between the two solutions. If such a collapse scenario is to be called a solution of the field equations, the two Vaidya spacetimes separated by the CH (null hypersurface layer) must form a smooth solution. Hence, the thin null shell with the stress energy should be matched with two spacetimes before and after. Barrabès and Israel (and Poisson) have analyzed in detail the conditions for immersion of such null surface layers between two general spherically symmetric spacetimes. To implement our model we follow the prescription of matching across null hypersurface by Barrabès and Israel [23] (see also Poisson [24]).

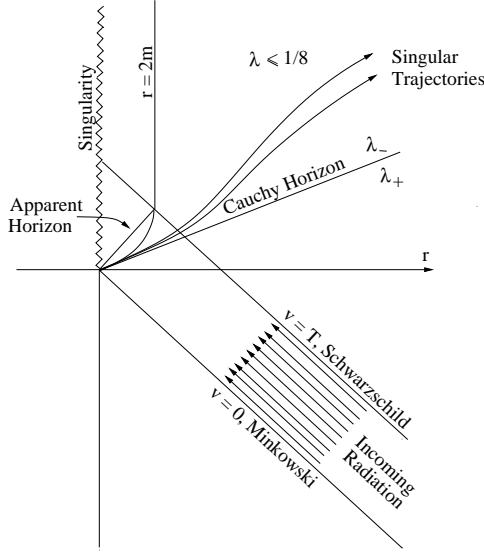


FIG. 1: Naked singularity forming in the radiation collapse

Let the two spacetimes separating the first singular light ray (CH) be given by  $\lambda_+$  before and  $\lambda_-$  after (see Fig 1). We can describe the spacetime metric across CH in the following form

$$ds_I^2 = -(1 - \lambda_+ x_+) du_+^2 + 2du_+ dr + r^2 d\Omega^2 \quad (6)$$

$$ds_{II}^2 = -(1 - \lambda_- x_-) du_-^2 + 2du_- dr + r^2 d\Omega^2 \quad (7)$$

where  $x_+ = u_+/r$  and  $x_- = u_-/r$ . Here region I and region II correspond to spacetime before and after formation of Cauchy horizon, respectively. In order to glue these two Vaidya spacetimes along the null hypersurface  $\Sigma$  (CH) we should have

$$x_+ = \gamma_+ = \frac{1}{2\lambda_+} \left[ 1 - \sqrt{1 - 8\lambda_+} \right], \quad \frac{du_+}{dr} = \frac{2}{(1 - \lambda_+ x_+)} \Big|_{\Sigma}, \quad (8)$$

in spacetime I, and

$$x_- = \gamma_- = \frac{1}{2\lambda_-} \left[ 1 - \sqrt{1 - 8\lambda_-} \right], \quad \frac{du_-}{dr} = \frac{2}{(1 - \lambda_- x_-)} \Big|_{\Sigma} \quad (9)$$

in spacetime II. On the boundary we have from continuity

$$u_+|_{\Sigma} = \frac{\gamma_+}{\gamma_-} u_-|_{\Sigma}, \quad (10)$$

Defining tangent vectors on the Cauchy horizon :

$$k^a \equiv e_{(1)}^a = \left[ \frac{2r}{(r - \lambda_+ u_+)}, 1, 0, 0 \right] \Big|_{\Sigma}, \text{ region I}$$

$$k^a \equiv e_{(1)}^a = \left[ \frac{2r}{(r - \lambda_- u_-)}, 1, 0, 0 \right] \Big|_{\Sigma}, \text{ region II}$$

and

$$e_{(2)}^a = \frac{\partial}{\partial \theta}, \quad e_{(3)}^a = \frac{\partial}{\partial \phi}. \quad (11)$$

for region I & II. Where  $y^\alpha = (r, \theta, \phi)$  are the intrinsic coordinates on  $\Sigma$  ( $\alpha = 1, 2, 3$ ), and we take  $r$  to be the parameter of the null generator. The transverse vectors completing the basis for region I and II are given by

$$N^a = \left[ 0, -\frac{1}{2} \left( 1 - \frac{\lambda_+ u_+}{r} \right), 0, 0 \right] \Big|_{\Sigma}, \text{ region I}$$

$$N^a = \left[ 0, -\frac{1}{2} \left( 1 - \frac{\lambda_- u_-}{r} \right), 0, 0 \right] \Big|_{\Sigma}, \text{ region II}$$

satisfying

$$N_a N^a = 0, \quad N_a k^a = -1, \quad N_a e_{(A)}^a = 0, \quad (12)$$

and where  $(A) = \{\theta, \phi\}$ . The transverse curvature  $C_{AB}^\pm$ , and the intrinsic metric of the surface layer ( $\sigma_{AB}$ ) is given by

$$\sigma_{AB} dx^A dx^B = r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (13)$$

and

$$C_{AB} = -N_\alpha e_{(A);\beta}^\alpha e_{(B)}^\beta. \quad (14)$$

We find the surface energy density and pressure of the null layer for Vaidya case as

$$\mu = \sigma^{AB} C_{AB} = \frac{[M]}{4\pi r^2} = \frac{\lambda_- u_- - \lambda_+ u_+}{4\pi r^2}, \quad p = [h(\lambda, M)]. \quad (15)$$

First we note that  $h(\lambda, M)$  quantifies jump in pressure across the CH. Since it is transverse component to Cauchy surface it does not affect the physics of energy propagating along the CH which is of interest to us here. If energy is transported along the CH the energy density

$\mu$ , of the null layer must be positive definite. It follows from the continuity of the boundary layer from Eqs. (8) and (9) that

$$\frac{[M]}{4\pi r^2} = \frac{u_+}{\gamma_+} \left[ \frac{\sqrt{1-8\lambda_+} - \sqrt{1-8\lambda_-}}{8\pi r^2} \right].$$

Hence, CH can carry energy to either a local or a distant observer. Therefore, as a result the rate of collapse slows down ( $\lambda_- < \lambda_+$ ), which results in a net positive energy density on the CH. Furthermore, this surface energy on the CH has a clear physical interpretation. To see this consider the motion of a freely falling timelike observer (four velocity  $u^a$ ,  $u^a u_a = -1$ ) in Vaidya spacetime given in Eq. (1) (analysis of timelike trajectories in Vaidya spacetime has been worked out [21]).

$$u^a \equiv e_{(1)}^a = \left[ \frac{P}{r}, \frac{(1-\lambda x)P}{2r} - \frac{r}{2P}, 0, 0 \right], \text{ region I}$$

$$P = \frac{(c-s) \pm \sqrt{(c-s)^2 + r^2 x(2 + \lambda x^2 - x)}}{(2 + \lambda x^2 - x)}, \text{ region II}$$

Where  $c$  is a constant labeling different timelike geodesics and  $s$  is the affine parameter. Positive sign solutions terminate at the singularity  $r = u = 0$  with a positive definite tangent  $x = \beta$  and hence do not intersect the CH. For all timelike radial observers intersecting the CH we have

$$u_a k^a = \frac{r}{(1-\lambda x)P}, \quad (16)$$

and therefore at the cauchy horizon we have

$$[u_a k^a] = [u_a e_{(A)}^a] = 0. \quad (17)$$

It follows that in imploding null dust collapse CH can be immersed between two Vaidya spacetimes (with linear mass function) with the parameter  $\lambda_+ \leq \lambda_-$ . In case when  $\lambda_+ = \lambda_-$  the matching across  $\Sigma$  is smooth and no energy is carried along the first ray. In the case otherwise the rate of collapse slows down and facilitates the positive surface density on the null boundary.

It has been shown that there is a family of out going geodesics which terminate in the past at the singularity with a definite tangent  $x = \beta > \gamma$ . The path of such outgoing null geodesics has been calculated earlier (see [22]) and is given by Eq. (5). The problem can be considerably simplified if we can write the metric in terms of out-going null geodesics. In this representation CH corresponds to a constant value of one of the coordinates. Let us consider a general spherically symmetric space-time  $M^\pm$  given by

$$ds^2 = -e^{2\psi} \left( 1 - \frac{2m}{r} \right) dV^2 + 2\zeta e^\psi dV dr + r^2 d\Omega^2. \quad (18)$$

Here  $\psi_\pm, m_\pm$  are functions of  $V_\pm$  and  $r$ . Null layers given by  $V_\pm = \text{constant}$  are out going if  $\zeta = -1$  and ingoing if  $\zeta = 1$ . The density and pressure of the null shell surface immersed in the two spacetimes is

$$\mu = \sigma^{AB} C_{AB} = -\zeta \frac{[m]}{4\pi r^2},$$

$$p = -\zeta \frac{1}{8\pi} \frac{\partial \psi}{\partial r}. \quad (19)$$

In order to analyse the case of family of null geodesics let us consider a coordinate transformation for the space-time given in equation (1)  $v \rightarrow V, r \rightarrow r$ . The Vaidya metric the for spacetime  $M_\pm$  now becomes

$$ds^2 = -e^\psi \left[ e^\psi \left( 1 - \frac{2m(V, r)}{r} \right) dV^2 + 2dV dr \right] + r^2 d\Omega^2,$$

with metric function  $\exp(\psi)$

$$e^{\psi_\pm} = \frac{r\lambda_\pm(\beta_\pm - x_\pm)(x_\pm - \gamma_\pm)}{V_\pm(1 - \lambda_\pm x_\pm)}, \quad (20)$$

where  $2m_\pm(V_\pm, r) = \lambda_\pm v_\pm(V_\pm, r)$  and  $\psi_\pm = \psi_\pm(V_\pm, r)$ . Here  $V_\pm = \text{constant}$  are outgoing singular geodesics with normal  $k^a = \delta_r^a$ . Hence outgoing singular null layers immersed between the two Vaidya spacetimes with different mass functions ( $\lambda_+ < \lambda_- < \frac{1}{8}$ ) have non vanishing surface density  $\mu$  and  $p$  allowing energy to escape. Though for lightlike shells there is no rest frame and therefore  $\mu$  and  $P$  cannot be given an absolute meaning as surface density and pressure nonetheless as rightly pointed out by Israel they serve perfectly well to determine the results of measurement by any observer. In this regard as shown by Israel that for a radially freely falling observer momentum normal to the shell is continuous and the energy density associated with the shell as measured by this radially freely falling observer ( $u^a = dx^a/d\tau = [\dot{u}, \dot{r}, 0, 0]$ ) is given by

$$T_\Sigma^{ab} u_a u_b = \frac{[m]}{4\pi r^2} \delta(\tau) (k^a u_a) \quad (21)$$

and is accompanied by equal energy flux. Here  $\tau = 0$  is the equation of  $\Sigma$ .

We briefly consider now another scenario, namely the inhomogeneous dust collapse. The metric describing a spherically symmetric space-time is given by

$$ds^2 = -dt^2 + \frac{R'^2}{1+f} dr^2 + R^2 d\Omega^2, \quad (22)$$

where  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ , and  $R = R(t, r)$  and  $f = f(t, r)$  are arbitrary functions of  $t$  and  $r$ . The metric in (22) has to satisfy field equations which can be put in a form

$$\dot{R} = -\sqrt{f + \frac{F}{R}} \quad (23)$$

The functions,  $F = F(r)$ ,  $R = R(t, r)$  and  $f(r) > -1$  are  $C^2$  functions throughout the cloud. Notation  $(\dot{\phantom{a}})$  and  $(\phantom{a})'$

are used to denote partial differentiation with respect to  $r$  and  $t$ . Consider marginally bound case  $f(r) = 0$ , and  $F(r) = F_0 r$ . Existence of naked singularity in this case ( $\beta < 3$ ) is well established (see [11] and ref. therein). Function  $F = F(r)$  is interpreted as the mass function and for physical reasons  $F(r) \geq 0$ ,  $F'(r) \geq 0$ , and gives mass enclosed in a given shell of co-moving radius  $r$ . The CH is a null ray  $R = x_0 r$  where  $x = x_0$  is lowest of the real and positive root of the algebraic equation

$$2x^4 + x^3\sqrt{F_0} - 2x + 2\sqrt{F_0} = 0 \quad (24)$$

The two dust spacetimes  $M_{\pm}$  separating the first singular light ray (CH) be given by respective mass functions,

$F_{-}(r) = F_0 r$  before, and  $F_{+}(r) = rP^2(r)$  after, where  $P(r)$  satisfies

$$P(a + bP)^c = F_1 r, \quad (25)$$

where  $F_1$  is a constant and coefficients can be determined as  $a = 2x_0^{3/2}$ ,  $b = (x_0^{3/2} + 2)/(x_0^{3/2} - 1)$  and  $c = -3x_0^{3/2}/(x_0^{3/2} + 2)$ .

An argument similar to one for Vaidya model shows that that cauchy horizon can be immersed between these two two different dust solutions with a non vanishing surface energy density given by different values of constants  $F_1$  and  $F_0$ .

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